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Scaling Limit and Critical Exponents for Two-Dimensional Bootstrap Percolation

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Consider a cellular automaton with state space $\{0,1\}^{\mathbb{Z}^2}$ where the initial configuration ω_0 is chosen according to a Bernoulli product measure, 1's are stable, and 0's become 1's if they are surrounded by at least three neighboring 1's. In this paper we show that the configuration ω_n at time n converges exponentially fast to a final configuration $\bar{\omega}$, and that the limiting measure corresponding to $\bar{\omega}$ is in the universality class of Bernoulli (independent) percolation. More precisely, assuming the existence of the critical exponents β, η, ν and γ, and of the continuum scaling limit of crossing probabilities for independent site percolation on the close-packed version of $\mathbb{Z}^2$ (i.e. for independent $*$-percolation on $\mathbb{Z}^2$), we prove that the bootstrapped percolation model has the same scaling limit and critical exponents. This type of bootstrap percolation can be seen as a paradigm for a class of cellular automata whose evolution is given, at each time step, by a monotonic and nonessential enhancement [Aizenman and Grimmett, *J. Stat. Phys*. **63**: 817–835 (1991); Grimmett, *Percolation*, 2nd Ed. (Springer, Berlin, 1999)].

KEY WORDS: Bootstrap percolation; scaling limit; critical exponents; universality.

1. INTRODUCTION AND MOTIVATIONS

Bootstrap percolation is a cellular automaton with state space $\{0,1\}^{\mathbb{Z}^d}$, which evolves in discrete time according to the following rule: a given configuration of 0's and 1's is updated by changing to 1 each 0 with at least l neighboring 1's and leaving the rest of the configuration unchanged. Here l is a nonnegative integer no bigger than $2d$, and the initial configuration

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is chosen according to a Bernoulli product measure with parameter p (the initial density of 1's).

It is known from the work of van Enter[17] and Schonmann[14] that if $l \leqslant d$, then almost all initial configurations evolve toward the constant configuration with 1's at all sites. On the other hand, it is clear that when $l > d$, the 1's do not take over completely for almost every initial configuration. For example, if $l = 2d$, the only 0's that become 1's are those completely surrounded by 1's. The configuration changes only once and the final measure is in some sense very close to a product measure.

The case $l = 2d - 1$ is already much more interesting; it is studied in ref. 7, where the limiting measure (whose existence is ensured by the monotonicity of the dynamics) is shown to have exponentially decaying correlations, and the density function to be analytic in [0, 1] (for simplicity, the authors restrict their attention to $d = 2$, but all arguments used are immediately seen to hold qualitatively for any $d \geqslant 2$). In this paper, we consider the same model studied in ref. 7, with $l = 2d - 1$ and $d = 2$.

The exponential decay of correlations proved in ref. 7 shows that the bootstrap dynamics generates only short-range correlations between different sites. It is an open question, in general, whether introducing short-range correlations modifies the critical exponents and the continuum scaling limit (see Section 1.1). Based on very general renormalization group arguments, the answer to this question is expected to be negative under a broad class of conditions (see, for example, ref. 6), but very few rigorous results are available, especially below the *upper critical dimension*, where the values of the critical exponents are expected to be different from those predicted by mean-field theory (there are, however, some exceptions – see, e.g., refs. 2–4, and 12). The main goal of this paper is to present a model for which this question can be answered rigorously.

Our first result, Theorem 1, states that the probability $\Pi(n)$ that the origin changes state after time n decays exponentially in n. Following its proof, we present a proof of the exponential decay of correlations, Theorem 2, which is somewhat different than that of ref. 7. The purpose of our proof is to show that the same mechanism is responsible for the exponential decay of correlations and the exponential convergence to the final configuration (Theorem 1).

The mechanism we are referring to has to do with the nature of the bootstrap dynamics (which only removes "dangling ends" from clusters of 0's, starting from the tip) combined with the fact that large "tree-like" clusters are "unlikely" in Bernoulli percolation. These two observations are sufficient to prove the results mentioned above, and also imply that the bootstrap dynamics only removes relatively small pieces of clusters of 0's, leaving unchanged the large pieces that are relevant in the scaling limit. This is the reason why the bootstrapped measure has the same critical exponents and scaling limit as independent percolation. In this (very special) case, therefore, the reason for the "universal" behavior (see Section 1.1) of the bootstrap model is quite clear, without having to resort to more general (non-rigorous) renormalization group arguments.

The main results are presented in Section 3, and the proofs given in Section 4. The proofs are based on ideas developed in refs. 2 and 3, and the main tool is the natural coupling between independent and bootstrap percolation provided by the bootstrap dynamics itself, which allows to draw conclusions regarding the bootstrap model by estimating the probability of events under the initial product measure. In particular, the coupling allows to compare bootstrap and Bernoulli percolation and show that they do not differ "macroscopically."

Besides being interesting in its own right, the bootstrap dynamics considered in this paper can also be seen as a particular example of a special class of cellular automata on various lattices whose evolution is given, at each time step, by a *monotonic* and *nonessential enhancement* of *finite range* (see refs. 1 and 10 for the relevant definitions). Such cellular automata would be called *subcritical* in the language of ref. 9 (although they do not represent all subcritical cellular automata).

In order to extend the results of the present paper to the whole class of cellular automata specified above, one needs first of all to find candidates for the protected sites of Definition 2.1. The existence of suitable candidates for that role is not obvious in that generality, but can be proved using results obtained in a paper in preparation by the author. Once this is done, the proofs of the main theorems would proceed in much the same way as in this paper.

1.1. Universality

Our main motivation for studying the type of questions addressed here (see also refs. 2–4) is related to the idea of universality, according to which most statistical-mechanical systems fall into universality classes such that systems belonging to the same class have the same critical exponents (the exponents describing the nature of the divergence of certain quantities or their derivatives near or at the critical point, where a second order phase transition occurs).

A closely related notion of universality has to do with the continuum scaling limit, a limit in which the *microscopic* scale of the system (e.g., the lattice spacing for systems defined on a lattice) is sent to zero, while focus is kept on features manifested on a *macroscopic* scale. Such a limit is only meaningful at the critical point, where the correlation length (i.e., the "natural length scale" of the system) is supposed to diverge. It seems that universality in terms of the scaling limit is a stronger notion than that in terms of critical exponents. In ref. 16, some knowledge of the scaling limit is used to determine critical exponents in the case of two-dimensional independent site percolation on the triangular lattice, but there is no general result in that direction.

The concept of universality and the existence of universality classes arise naturally in the theory of critical phenomena based on the renormalization group, and are backed by strong theoretical and experimental evidence. Below the upper critical dimension, however, only few rigorous results are available.

2. DEFINITION OF THE MODEL AND PRELIMINARY RESULTS

Consider a bootstrap percolation model on $\mathbb{Z}^2$ with initial configuration $\omega = \{\omega(x)\}_{x\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}$ chosen according to a product $P_p = \Pi_{x\in\mathbb{Z}^2} \nu_x$ of Bernoulli measures $\{\nu_x\}_{x\in\mathbb{Z}^2}$ with parameter p (i.e., $\nu_x[\omega(x)=1] = p = 1 - \nu_x[\omega(x)=0]$); E_p will denote expectation with respect to P_p. The evolution is given by the following rules:

- updates are performed at discrete times $n = 1, 2, \ldots$
- 1's are stable,
- at the next update, a 0 becomes 1 if it has at least three neighboring 1's.

Given an initial configuration ω, the bootstrapped configuration is denoted by $\bar{\omega}$ and the limiting distribution by $\bar{P}_p$. We will call the sites of $\mathbb{Z}^2$ *open* if they are assigned value 1 and *closed* if they are assigned value 0. Given a subset D of $\mathbb{Z}^2$, we denote $|D|$ its cardinality and by ω_D the configuration ω restricted to D. A subset $D \in \mathbb{Z}^2$ is called a *plaquette* if it is composed of four sites which are the vertices of a square of side length 1.

We denote by p_c the critical value of independent site percolation on $\mathbb{Z}^2$ and by $p_c^* = 1 - p_c$ the critical value of independent $*$-percolation on the same lattice, which corresponds to site percolation on $\mathbb{Z}^2_{cp}$, the *close-packed* version of $\mathbb{Z}^2$ (obtained by adding the diagonals to each face of $\mathbb{Z}^2$). We call $\mathbb{Z}^2$-*path* (resp. $*$-*path*) an ordered sequence $(x_0, \ldots, x_k)$ of sites of $\mathbb{Z}^2$ such that x_{i-1} and x_i are neighbors in $\mathbb{Z}^2$ (resp. in $\mathbb{Z}^2_{cp}$) for $i = 1, \ldots, k$ and $x_i \neq x_j$ for $i \neq j$. A $\mathbb{Z}^2$-*loop* (resp. $*$-*loop*) is a $\mathbb{Z}^2$-path (resp., $*$-path) that ends at a $\mathbb{Z}^2$-neighbor (resp., $*$-neighbor) of the starting site. A path or a loop will be called closed or open if all its sites are closed or open, respectively. We call *length* of a path or loop the number of sites in it.

Definition 2.1. A closed site $x \in \mathbb{Z}^2$ is called *stable* if and only if $\bar{\omega}(x) = 0$. A site is said to be *protected* if it is closed and is part of a group of four closed sites forming a plaquette.

Clearly, a protected site is stable, together with the other three sites that complete the plaquette of Definition 2.1, since each one of them has (at least) two closed $\mathbb{Z}^2$-neighbors.

The following are two elementary but useful lemmas.

Lemma 2.1. If x and y are stable closed sites and ω contains a closed $\mathbb{Z}^2$-path π joining x and y, then all the sites in π are stable. Closed $\mathbb{Z}^2$-loops are also stable.

Proof. For the first claim, it is enough to observe that each site in π other than x or y has at least two closed $\mathbb{Z}^2$-neighbors in ω. In a $\mathbb{Z}^2$-loop, every site has at least two closed $\mathbb{Z}^2$-neighbors. ∎

For (x, x') an ordered pair of neighbors in $\mathbb{Z}^2$, we define the *partial cluster* $C_{(x,x')}$ to be the set of sites $y \in \mathbb{Z}^2$ such that there is a $\mathbb{Z}^2$-path $(x_0 = x', x_1, \dots, x_k = y)$, with $x_1 \neq x$, whose sites are all open or all closed.

Lemma 2.2. A closed $\mathbb{Z}^2$-path $(y_0, \dots, y_k)$ in ω is stable (i.e., all its sites are stable) if $C_{(y_1,y_0)}$ and $C_{(y_{k-1},y_k)}$ both contain protected sites.

Proof. The path $(y_0, \dots, y_k)$ in ω is stable because there exists a (generally longer) closed path that starts and ends at stable sites and contains $(y_0, \dots, y_k)$ as a subpath. Since the starting and ending sites of such a path are stable, all the other sites of the path, including $y_0, \dots, y_k$, are also stable by an application of Lemma 2.1. ∎

We will denote by ω_n the percolation configuration at time n, i.e., after n updates of the initial configuration. With this notation we have $\omega_0 = \omega$ (the initial configuration) and $\omega_\infty = \bar{\omega}$ (the final configuration). Our first result concerns the speed of convergence of ω_n to $\bar{\omega}$.

Theorem 1. Let $\Pi(n)$ be the probability that the origin changes state after time n. Then, for each $p \in [0, 1]$ there exists $c_0 > 0$ such that $\Pi(n) \leqslant \exp(-c_0 n)$.

Proof. Let o denote the origin of $\mathbb{Z}^2$. If $\omega_0(o) = 1$, the origin never changes state, therefore we will assume, without loss of generality, that $\omega_0(o) = 0$ and also that $0 < p < 1$. To analyze when the origin becomes 1, we consider its cluster C_o at time 0. Let x_1, x_2, x_3, x_4

be the four $\mathbb{Z}^2$-neighbors of the origin in some deterministic order. For x_i, $i=1,2,3,4$, we define the *branch* C_i to be the partial cluster $C_{(o,x_i)}$. If $\omega_0(x_i)=1$, we say that C_i is *empty*.

Our first observation is that if the branches C_i, $i=1,2,3,4$, are not distinct, the origin belongs to a $\mathbb{Z}^2$-loop and is stable by Lemma 2.1. We also notice that, for the origin to become 1, no more than one branch C_i can have a stable site, otherwise the origin would again be stable by Lemma 2.1. We will then assume that the branches C_i, $i=1,2,3,4$, are distinct, and that at most one of them contains a stable site. Notice that the branches that do not contain stable sites have a tree-like structure (they do not contain $\mathbb{Z}^2$-loops).

Consider first the case in which exactly one branch contains a stable site. The origin will then become 1 at some time n equal to one plus the length of a longest self-avoiding $\mathbb{Z}^2$-path contained in one of the remaining branches. If no branch contains a stable site, let C_j be a branch containing a longest $\mathbb{Z}^2$-path and π be a longest $\mathbb{Z}^2$-path not contained in C_j. Then the origin will become 1 at some time n equal to one plus the length of π.

The discussion above shows that a necessary condition for the origin to change state after time n is that at least one of the four branches C_i contains a path of length at least n *and* no stable site. Since a protected site is stable, to complete the proof, it suffices to show that there are $\alpha>0$ and $K<\infty$ such that

$$P_p(|C_i|\geqslant n \text{ and } C_i \text{ contains no protected site})\leqslant K\, e^{-\alpha n}. \tag{1}$$

To prove (1), we partition $\mathbb{Z}^2$ into disjoint plaquettes and denote by S the collection of these plaquettes. We do an algorithmic construction of C_i (as in, e.g., ref. 8), where the order of checking the state of sites is such that when the first site in a square from S is checked and found to be closed, then the other three sites in that plaquette are checked next. Then standard arguments show that the probability in (1) is bounded above by $K\,[1-(1-p)^4]^{(n/4)}$. ∎

Remark 2.1. We note that one can improve Theorem 1, namely prove exponential convergence *uniformly* in $p\in[0,1]$ (i.e., it is possible to get a constant $c_0>0$ independent of p). This is done by using the proof given above for values of p smaller than some $p_0>p_c^*$, together with the fact that for $p\geqslant p_0$ the size of the closed cluster of the origin at time 0 has an exponential tail.[10] (We have chosen to give the argument in the proof simply because it has the advantage of being valid for all values of p.)

Using arguments analogous to those in the proof of Theorem 1, one can get exponential decay of correlations for $\bar{P}_p$, which was proved, in a somewhat different way, in ref. 7. This result is important in the context of the present paper because it suggests (see, for example, ref. 6) that $\bar{P}_p$ is in the universality class of independent percolation, as we will show in the next section. We include here a proof of the result (see ref. 7 for the original proof) in order to show how the same mechanism is responsible for the exponential decay of correlations of the limiting measure and the exponential convergence to the final configuration (Theorem 1). As it will be clear from the proofs of the main results, that same mechanism is also responsible for the fact that $\bar{P}_p$ is in the universality class of independent percolation. Such mechanism explains, in this particular case, the model's "universal" behavior and its relation with the exponential decay of correlations.

For $x \in \mathbb{Z}^2$, let $\mathrm{d}(o, x)$ be one plus the number of sites between o and x along a shortest $\mathbb{Z}^2$-path from o to x, and let $B_x(r) = \{y \in \mathbb{Z}^2 : \mathrm{d}(y, x) < r\}$.

Theorem 2. (ref. 7). $\bar{P}_p$ has exponentially decaying correlations

$$|E_p[\bar{\omega}(o)\bar{\omega}(x)] - E_p[\bar{\omega}(o)]E_p[\bar{\omega}(x)]| \leqslant R \exp[-c_0' \, d(o, x)], \tag{2}$$

where $R < \infty$ and $c_0' > 0$.

Proof. Denote by $A_x(n)$ the event that $\bar{\omega}(x)$ is determined only by the configuration $\omega_{B_x(n)}$ inside $B_x(n)$, and by $A_x^c(n)$ its complement. The proof rests on the observation that if $\mathrm{d}(o, x) > 2n$, then conditioned on $A_o(n)$ and $A_x(n)$, the random variables $\bar{\omega}(o)$ and $\bar{\omega}(x)$ are independent.

Before proceeding with the proof, we notice that a necessary condition for $A_o^c(n)$ to occur is that the origin be closed at time 0 and that there be at least one branch C_i of the cluster of the origin at time 0 that reaches the boundary of $B_o(n)$ and has no stable site inside $B_o(n)$. This event is analogous to the one considered at the end of the proof of Theorem 1. Then, arguments analogous to those used there to get (1) give the bound

$$P_p[A_o^c(n)] \leqslant \exp(-\alpha' n) \tag{3}$$

for some $\alpha' > 0$.

Take N such that $P_p[A_o^c(N)] < 1/2$ and consider the set of sites $\{x \in \mathbb{Z}^2 : \mathrm{d}(o, x) \geqslant 3N\} = \mathbb{Z}^2 \setminus B(3N)$. For a site in $\mathbb{Z}^2 \setminus B(3N)$, we take

$n = \lceil \mathrm{d}(o,x)/3 \rceil$ and write, thanks to the observation above,

$$\begin{aligned} E_p[\bar{\omega}(o)\,\bar{\omega}(x)] &= E_p[\bar{\omega}(o)\,\bar{\omega}(x) \mid A_o(n) \cap A_x(n)]\,\{1 - P_p[A_o^c(n)]\}^2 \\ &\quad + E_p[\bar{\omega}(o)\,\bar{\omega}(x) \mid A_o^c(n) \cup A_x^c(n)]\,\{2 - P_p[A_o^c(n)]\}\, P_p[A_o^c(n)], \qquad (4) \\ &= E_p[\bar{\omega}(o) \mid A_o(n) \cap A_x(n)]\, E_p[\bar{\omega}(x) \mid A_o(n) \cap A_x(n)]\,\{1 - P_p[A_o^c(n)]\}^2 \\ &\quad + E_p[\bar{\omega}(o)\,\bar{\omega}(x) \mid A_o^c(n) \cup A_o^c(n)]\,\{2 - P_p[A_o^c(n)]\}\, P_p[A_o^c(n)], \qquad (5) \end{aligned}$$

where we have used

$$P_p[A_o(n) \cap A_x(n)] = P_p[A_o(n)] P_p[A_x(n)] = \{1 - P_p[A_o^c(n)]\}^2 \tag{6}$$

and

$$P_p[A_o^c(n) \cup A_x^c(n)] = 1 - P_p[A_o(n) \cap A_x(n)] = \{2 - P_p[A_o^c(n)]\}\, P_p[A_o^c(n)], \tag{7}$$

which follow from the observation that $A_o(n)$ and $A_x(n)$ are independent events because $\mathrm{d}(o,x) > 2n$.

We now write

$$E_p[\bar{\omega}(o) \mid A_o(n) \cap A_x(n)] = \frac{E_p[\bar{\omega}(o)] - E_p[\bar{\omega}(o) \mid A_o^c(n) \cup A_x^c(n)]\,\{2 - P_p[A_o^c(n)]\}\, P_p[A_o^c(n)]}{\{1 - P_p[A_o^c(n)]\}^2} \tag{8}$$

and the same for $E_p[\bar{\omega}(x) \mid A_o(n) \cap A_x(n)]$, and plug the two expressions in (5) to get

$$\begin{aligned} E_p[\bar{\omega}(o)\bar{\omega}(x)] &= \frac{1}{\{1 - P_p[A_o^c(n)]\}^2} \times E_p[\bar{\omega}(o)]\, E_p[\bar{\omega}(x)] \\ &\quad + R_1\, P_p[A_o^c(n)] + R_2\, P_p[A_o^c(n)]^2 \end{aligned} \tag{9}$$

for some constants R_1 and R_2. From (3) and (9), we immediately see that

$$|E_p(\bar{\omega}(0)\,\bar{\omega}(x)) - E_p(\bar{\omega}(o))\, E_p(\bar{\omega}(x))| \leqslant R\, P_p[A_o^c(n)] \leqslant R\, e^{-cn} \tag{10}$$

for some $R < \infty$ and $c > 0$.

For the sites in $\mathbb{Z}^2 \setminus B(3N)$, the proof is concluded by taking $c_0' = c/3$. For the sites in $B(3N)$, we just have to choose a constant R large enough so that $R \exp(-3Nc_0') \geqslant 1$. ∎

Remark 2.2. We note that one can get exponential decay of correlations *uniformly* in $p \in [0, 1]$ (as in ref. 7) using the fact that for $p > p_c^*$ the size of the closed cluster of the origin at time 0 has an exponential tail[10] (see Remark 2.1).

We conclude this section with Proposition 2.1, which identifies the critical density of our bootstrap percolation model on $\mathbb{Z}^2_{cp}$ with p_c^*, showing that the bootstrapping rule employed here does not shift the critical point. This motivates the next section, where we analyze the continuum scaling limit of crossing probabilities and some critical exponents of the bootstrapped model on $\mathbb{Z}^2_{cp}$ when the initial density of 1's is p_c^*.

Proposition 2.1. The following results hold for $\bar{\omega}$:

1.(i) Closed sites do not percolate if $p > p_c^*$ and percolate if $p < p_c^*$.

(ii) If $p = p_c^*$, closed sites do not percolate and the mean cluster size for the closed component is infinite.

2.(i) Open sites do not $*$-percolate if $p < p_c^*$ and $*$-percolate if $p > p_c^*$.

(ii) If $p = p_c^*$, open sites do not $*$-percolate and the mean $*$-cluster size for the open component is infinite.

Proof. Let us begin with the proofs of 1.(i) and 2.(i), which are elementary. If $p > p_c^*$, closed sites do not percolate in ω, that is before bootstrapping the open sites, and therefore cannot possibly percolate in $\bar{\omega}$, after bootstrapping the open sites. If $p < p_c^*$, on the contrary, closed sites do percolate in ω, and since any doubly-infinite closed $\mathbb{Z}^2$-path (i.e., a closed $\mathbb{Z}^2$-path that can be split in two disjoint infinite paths) contained in ω is stable and therefore it is also contained in $\bar{\omega}$, this implies that closed sites percolate in $\bar{\omega}$ and concludes the proof of 1.(i).

To prove 2.(i), it suffices to notice that for $p < p_c^*$, closed sites percolate in ω and the origin is surrounded by infinitely many $\mathbb{Z}^2$-loops of closed sites. Such closed loops are stable and therefore still exist in $\bar{\omega}$ and prevent open sites from $*$-percolating. On the other hand, if $p > p_c^*$ open sites $*$-percolate already in ω, which concludes the proof of 2.(i).

1.(ii) and 2.(ii) can be proved together using a theorem of Russo.[13] At $p = p_c^*$, in ω the origin is surrounded by infinitely many $\mathbb{Z}^2$-loops of closed sites and infinitely many $*$-loops of open sites. Both types of loops are stable and therefore in $\bar{\omega}$ there is no percolation of closed sites, nor $*$-percolation of open sites. By an application of a theorem of Russo,[13] this implies that both the mean cluster size of the closed component and the mean $*$-cluster size of the open component diverge. ∎

3. MAIN RESULTS

In this section, we present the main results of this paper; the proofs will be given in Section 4. The results presented in this section hold for all the measures that are intermediate between the initial measure P_p and the limiting one $\bar{P}_p$. These form a one parameter family $\{P_{p,n}\}_{n\in\mathbb{N}}$ of measures, parametrized by time $n=1,2,3,\ldots$, and are increasingly different from P_p as n becomes larger.

3.1. The Continuum Scaling Limit of Crossing Probabilities

We take a "mesh" δ and consider the "scaling limit" of crossing probabilities for the percolation model $\bar{\omega}$ on $\delta\mathbb{Z}^2$ as $\delta\to 0$, focusing for simplicity on the probability of an open $*$-crossing of a rectangle aligned with the coordinate axes. A similar approach would work for any domain with a "regular" boundary, but it would imply dealing with more complex deformations of the boundary than that needed for proving the result for a rectangle.

Consider a finite rectangle $\mathcal{R}=\mathcal{R}(b,h)\equiv(-b/2,b/2)\times(-h/2,h/2)\subset\mathbb{R}^2$ centered at the origin of $\mathbb{Z}^2$, with sides of lengths b and h and aspect ratio $\rho=b/h$. We say that there is an open vertical $*$-crossing of $\mathcal{R}$ in ω (resp., $\bar{\omega}$) if $\mathcal{R}\cap\delta\mathbb{Z}^2$ contains a $*$-path of open sites from ω (resp., $\bar{\omega}$) joining the top and bottom sides of the rectangle $\mathcal{R}$, and call $\phi^*_\delta(b,h;n)$ the probability of such an open crossing at time n.

More precisely, there is a vertical open $*$-crossing at time n if there is a $*$-path $(x_0,x_1,\ldots,x_m,x_{m+1})$ in $\mathbb{Z}^2$ such that $\omega_n(x_j)=1$ for all j, $\delta x_0,\delta x_1,\ldots,\delta x_m,\delta x_{m+1}$ are all in $\mathcal{R}$, and the line segments $\overline{\delta x_0,\delta x_1}$ and $\overline{\delta x_m,\delta x_{m+1}}$ touch, respectively, the top side $[-b/2,b/2]\times\{h/2\}$ and the bottom side $[-b/2,b/2]\times\{-h/2\}$ of $\mathcal{R}$.

It is believed that the scaling limit of crossing probabilities for *independent* percolation exists and is given by Cardy's formula (see ref. 5 and 6); this has however been rigorously proved only for critical site percolation on the triangular lattice.[15] We will *assume* that $\lim_{\delta\to 0}\phi^*_\delta(b,h;0)=F(\rho)$, where F is a continuous function of its argument.

Theorem 3. Suppose that the scaling limit of the crossing probability of a rectangle $\mathcal{R}$ exists for independent critical site percolation on $\mathbb{Z}^2_{cp}$ and is given by a continuous function F of ρ. Then, the corresponding crossing probability in the bootstrapped model $\bar{\omega}$ with $p=p^*_c$ has the same scaling limit.

3.2. Critical Exponents

We will consider four percolation critical exponents, namely the exponents β (related to the percolation probability), ν (related to the correlation length), η (related to the connectivity function) and γ (related to the mean cluster size). The existence of these exponents has been recently proved,[11,16] and their predicted values confirmed rigorously, for the case of independent site percolation on the triangular lattice. Such exponents are believed to be universal for independent percolation in the sense that their value should depend only on the number of dimensions and not on the structure of the lattice or on the nature of the percolation model (e.g., whether it is site or bond percolation); that type of universality has not yet been proved.

Consider an independent percolation model with distribution P_p on a two-dimensional lattice $\mathbb{L}$ such that $0 < p_c < 1$. Let C_o be the open cluster containing the origin and $|C_o|$ its cardinality, then $\theta(p) = P_p(|C_o| = \infty)$ is the *percolation probability*. Arguments from theoretical physics suggest that $\theta(p)$ behaves roughly like $(p - p_c)^\beta$ as p approaches p_c from above.

It is also believed that the *connectivity function*

$$\tau_p(x) = P_p(\text{the origin and } x \text{ belong to the same cluster}) \tag{11}$$

behaves, for the Euclidean length $||x||$ large, like $||x||^{-\eta}$ if $p = p_c$, and like $\exp(-||x||/\xi(p))$ if $0 < p < p_c$, for some $\xi(p)$ satisfying $\xi(p) \to \infty$ as $p \uparrow p_c$. The *correlation length* $\xi(p)$ is defined by

$$\xi(p)^{-1} = \lim_{||x|| \to \infty} \left\{ -\frac{1}{||x||} \log \tau_p(x) \right\}. \tag{12}$$

$\xi(p)$ is supposed to behave like $(p_c - p)^{-\nu}$ as $p \uparrow p_c$. The *mean cluster size* $\chi(p) = E_p|C_o|$ is also believed to diverge with a power law behavior $(p_c - p)^{-\gamma}$ as $p \uparrow p_c$.

It is not clear how strong one may expect such asymptotic relations to be (for more details about critical exponents and *scaling theory* in percolation, see ref. 10 and references therein); for this reason the logarithmic relation is usually employed. This means that the previous conjectures are usually stated in the following form:

$$\lim_{p \downarrow p_c} \frac{\log \theta(p)}{\log(p - p_c)} = \beta, \tag{13}$$

$$\lim_{||x||\to\infty} \frac{\log\, \tau_{p_c}(x)}{\log ||x||} = -\eta, \tag{14}$$

$$\lim_{p\uparrow p_c} \frac{\log\, \xi(p)}{\log(p_c - p)} = -\nu, \tag{15}$$

$$\lim_{p\uparrow p_c} \frac{\log\, \chi(p)}{\log(p_c - p)} = -\gamma. \tag{16}$$

In the rest of the paper, $\theta(p)$, $\tau_p(x)$, $\xi(p)$ and $\chi(p)$ will indicate the percolation probability, connectivity function, correlation length and mean cluster size for independent site percolation on $\mathbb{Z}^2_{cp}$. For $n \in [1, \infty]$, let $\theta(p, n)$, $\tau_{p,n}(x)$, $\xi(p, n)$ and $\chi(p, n)$ be, respectively, the percolation probability, connectivity function, correlation length and mean cluster size on $\mathbb{Z}^2_{cp}$ for the bootstrapped model at time n, with $n = \infty$ corresponding to the fully bootstrapped configuration $\bar{\omega}$. The main theorem of this section is the following.

Theorem 4. There exist constants $0 < c_1, c_2 < \infty$ such that, $\forall n \in [1, \infty]$,

$$\theta(p) \leqslant \theta(p, n) \leqslant c_1\, \theta(p) \quad \text{for } p \in (p_c^*, 1], \tag{17}$$

$$\tau_p(x) \leqslant \tau_{p,n}(x) \leqslant p^{-c_2}\, \tau_p(x) \quad \text{for } p \in (0, p_c^*], \tag{18}$$

$$\xi(p, n) = \xi(p) \quad \text{for } p \in (0, p_c^*]. \tag{19}$$

The next corollary is an immediate consequence of Theorem 4 and its main application; it says that the bootstrapped percolation model (in fact, all models corresponding to n enhancements by bootstrapping, with $n \in [1, \infty]$) has the same critical exponents β, η, ν and γ as ordinary independent percolation.

Corollary 3.1. Suppose that the critical exponents β, η, ν and γ exist for independent site percolation on $\mathbb{Z}^2_{cp}$, then they also exist for the bootstrapped model and have for the latter the same numerical values as for the original model.

4. PROOFS OF THE MAIN RESULTS

In this section, we prove the main results of this paper, presented in Section 3.

4.1. Crossing Probabilities – Proof of Theorem 3

To prove the theorem, we need to compare the probability of an open vertical $*$-crossing of $\mathcal{R}$ in $\bar{\omega}$ with the probability of the same event in ω. In order to do that, we will use the natural coupling that exists between ω and $\bar{\omega}$ via bootstrapping. First of all notice that, if an open vertical $*$-crossing of $\mathcal{R}$ is present in ω, it is also present in ω_n, for all n, since open sites are stable. Therefore,

$$\lim_{\delta \to 0} \phi^*_\delta(b, h; n) \geqslant \lim_{\delta \to 0} \phi^*_\delta(b, h; 0) = F(\rho). \tag{20}$$

Equation (20) holds for all values of n, including $n = \infty$, so if we call $\bar{\phi}^*_\delta(b, h)$ the probability of an open vertical $*$-crossing of $\mathcal{R}$ from $\bar{\omega}$, we can write

$$\lim_{\delta \to 0} \bar{\phi}^*_\delta(b, h) \geqslant \lim_{\delta \to 0} \phi^*_\delta(b, h; 0) = F(\rho). \tag{21}$$

On the other hand, if an open vertical $*$-crossing of $\mathcal{R}$ is not present in ω, this implies the existence of a *closed horizontal* $\mathbb{Z}^2$*-crossing* of $\mathcal{R}$. For δ small such a crossing must involve many sites, and the probability of finding "near" its endpoints two sites x and y, belonging to the crossing, attached through closed $\mathbb{Z}^2$-paths to two stable closed sites x' and y' should be close to one. If such stable sites are found, Lemma 2.1 assures that at least the portion of the closed horizontal crossing from x to y is still present in $\bar{\omega}$. This suggests that, conditioned on having in ω a closed horizontal $\mathbb{Z}^2$-crossing of a slightly bigger (in the horizontal direction) rectangle, with high probability, in $\bar{\omega}$ there will be a closed horizontal $\mathbb{Z}^2$-crossing of $\mathcal{R}$ blocking any open vertical $*$-crossing. It is then enough to prove that this probability goes to one as $\delta \to 0$.

We will now make this more precise, adapting the proof of Theorem 1 of ref. 3. Consider the rectangle $\mathcal{R}' = \mathcal{R}(b', h)$ with b' slightly larger than b and aspect ratio $\rho' = b'/h$. It follows from our assumptions that

$$\phi^*(b', h; 0) \equiv \lim_{\delta \to 0} \phi^*_\delta(b', h; 0) = F(\rho') \tag{22}$$

and

$$\lim_{b' \to b} \phi^*(b', h; 0) = \lim_{\rho' \to \rho} F(\rho') = F(\rho). \tag{23}$$

If we now call $\phi_\delta(b', h+\delta; 0)$ the probability of a *closed horizontal* $\mathbb{Z}^2$*-crossing* of $\mathcal{R}(b', h+\delta)$ from ω, and $\phi_\delta(b, h+\delta; n)$ that of a *closed horizontal* $\mathbb{Z}^2$*-crossing* of $\mathcal{R}(b, h+\delta)$ from ω_n, the observation that a closed $\mathbb{Z}^2$-crossing can only be "eaten" from its endpoints yields

$$\phi_\delta(b, h+\delta; n) \geqslant \phi_\delta(b', h+\delta; 0), \tag{24}$$

as long as $b' > b$, and n is not too large (depending on $b'-b$ and δ).

Since a closed horizontal $\mathbb{Z}^2$-crossing of $\mathcal{R}(b, h+\delta)$ blocks any open vertical $*$-crossing of $\mathcal{R}(b, h)$ and vice versa, (24) yields

$$\phi^*_\delta(b, h; n) = [1 - \phi_\delta(b, h+\delta; n)] \leqslant [1 - \phi_\delta(b', h+\delta; 0)] = \phi^*_\delta(b', h; 0). \tag{25}$$

Keeping n fixed, we can let first δ go to zero and then b' go to b, thus obtaining from (25) a bound that, combined with (20), gives the desired result, at least for values of n that are not too large.

To complete the proof, we will extend (24) to all values of n, including $n = \infty$, at the cost of a correction that goes to zero as $\delta \to 0$. In order to do this, we will use Lemma 2.2 to show that if there is a closed horizontal crossing by $(y_0, \ldots, y_k)$ of $\mathcal{R}(b', h+\delta)$ at time 0, with high probability it does not "shrink" too much due to the effect of the dynamics, so that at all later times, including $n = \infty$, there is a closed horizontal crossing of $\mathcal{R}(b, h+\delta)$ by $(y_{k_1}, \ldots, y_{k_2})$. This is achieved by looking at the partial clusters containing the portions of $(y_0, \ldots, y_k)$ contained in $\mathcal{R}(b', h+\delta) \setminus \mathcal{R}(b, h+\delta)$ and searching for protected sites.

Noting that each of the partial paths $(y_0, \ldots, y_{k_1})$ and $(y_{k_2}, \ldots, y_k)$ contains of the order of $(b'-b)/\delta$ sites, we see that Lemma 2.2 implies that it suffices to show that there exist $\alpha > 0$ and $K < \infty$ such that for any deterministic (x, x'),

$$P_{p_c^*}(|C_{(x,x')}| \geqslant \ell \text{ and } C_{(x,x')} \text{ contains no protected site}) \leqslant K\, e^{-\alpha \ell}. \tag{26}$$

To prove (26), we proceed as in the proof of Theorem 1, that is, we partition $\mathbb{Z}^2$ into disjoint plaquettes and denote by S the collection of these plaquettes. We then do an algorithmic construction of $C_{(x,x')}$ where the order of checking the state of sites is such that when the first site in a plaquette from S is checked and found to be closed, then the other three sites in that plaquette are checked next. Again, standard arguments show that the probability in (26) is bounded above by $K\,[1-(1-p_c^*)^4]^{(\ell/4)}$. ∎

Remark 4.1. As already remarked, the proof of Theorem 3 shows that the result is valid for all the intermediate measures $P_{p_c^*, n}$.

4.2. Critical exponents

4.2.1. Proof of Theorem 4

For two subsets C and D of $\mathbb{Z}^2$, we denote by $\{C \longleftrightarrow D\}$ the event that some site in C is connected to some site in D by an open $*$-path, and by $\{C \longleftrightarrow \infty\}$ the event that some site in C belongs to an infinite open $*$-path.

The lower bound for $\theta(p,n)$ is obvious. For the upper bound, we let $\mathcal{N}_x^*$ be the set of $*$-neighbors of x and rely on the following observation. If no site in $\mathcal{N}_o^*$ belongs to an infinite open $*$-path at time 0, then the origin must be surrounded by a closed $\mathbb{Z}^2$-loop λ. It then follows, by Lemma 2.1, that each site in λ is stable. Therefore, the origin will not be connected to infinity by an open $*$-path at any later time. Thus,

$$\theta(p,n) \leqslant P_p(\mathcal{N}_o^* \longleftrightarrow \infty). \tag{27}$$

Since $\{o \longleftrightarrow \infty\}$ can be written as $\{\omega(o) = 1\} \cap \{\mathcal{N}_o^* \longleftrightarrow \infty\}$, and $\{\omega(o) = 1\}$ and $\{\mathcal{N}_o^* \longleftrightarrow \infty\}$ are independent at time 0,

$$P_p(o \longleftrightarrow \infty) = p\, P_p(\mathcal{N}_o^* \longleftrightarrow \infty). \tag{28}$$

From this we get

$$\theta(p,n) \leqslant p^{-1}\,\theta(p) \leqslant \frac{1}{p_c^*}\,\theta(p), \tag{29}$$

as required.

The lower bound for $\tau_{p,n}(x)$ is again obvious. To obtain the upper bound, we first note that for $||x||$ bounded, the inequality is trivial by choosing c_2 big enough so that the right-hand side of (18) exceeds 1. Next, for $\|x\|$ large enough, we notice that, unless $\{\mathcal{N}_o^* \longleftrightarrow \mathcal{N}_x^*\}$ at time 0, the origin and x must be separated by a closed $\mathbb{Z}^2$-loop surrounding one of them or by a doubly-infinite closed $\mathbb{Z}^2$-path, and therefore it cannot be the case that $\{0 \longleftrightarrow x\}$ at any later time. Thus,

$$\tau_{p,n}(x) \leqslant P_p(\mathcal{N}_o^* \longleftrightarrow \mathcal{N}_x^*). \tag{30}$$

Since $\{o \longleftrightarrow x\}$ can be written as $\{\omega(o) = \omega(x) = 1\} \cap \{\mathcal{N}_o^* \longleftrightarrow \mathcal{N}_x^*)\}$, and $\{\omega(o) = \omega(x) = 1\}$ and $\{\mathcal{N}_o^* \longleftrightarrow \mathcal{N}_x^*)\}$ are independent at time 0,

$$P_p(o \longleftrightarrow x) = p^2\, P_p(\mathcal{N}_o^* \longleftrightarrow \mathcal{N}_x^*). \tag{31}$$

From this we get

$$\tau_{p,n}(x) \leqslant p^{-2}\, \tau_p(x), \tag{32}$$

as required.

Equation (19) is an immediate consequence of (18) and the definition of $\xi(p)$; it is enough to observe that

$$\lim_{||x||\to\infty} \left\{ -\frac{1}{||x||} \left[\log \tau_p(x) - c_2 \log p \right] \right\} = \xi(p)^{-1}. \quad \blacksquare \tag{33}$$

4.2.2. Proof of Corllary 3.1

It follows from (17) and (18) that, for $p \in (p_c^*, 1]$ and $||x|| > 1$,

$$-\frac{\log \theta(p)}{\log (p - p_c^*)} \leqslant -\frac{\log \theta(p,n)}{\log (p - p_c^*)} \leqslant -\frac{\log \theta(p) + \log c_1}{\log (p - p_c^*)}, \tag{34}$$

$$\frac{\log \tau_{p_c^*}(x)}{\log \|x\|} \leqslant \frac{\log \tau_{p_c^*,n}(x)}{\log \|x\|} \leqslant \frac{\log \tau_{p_c^*}(x) - c_2 \log p_c^*}{\log \|x\|}. \tag{35}$$

For $p \in (0, p_c^*)$, observing that $\chi(p) = E_p \sum_{x\in\mathbb{Z}^2} I(o \longleftrightarrow x) = \sum_{x\in\mathbb{Z}^2} \tau_p(x)$ (where $I(\cdot)$ is the indicator function), (18) yields $\chi(p) \leqslant \chi(p,n) \leqslant p^{-c_2}\chi(p)$, and therefore

$$-\frac{\log \chi(p)}{\log (p - p_c^*)} \leqslant -\frac{\log \chi(p,n)}{\log (p - p_c^*)} \leqslant -\frac{\log \chi(p) - c_2 \log p}{\log (p - p_c^*)}. \tag{36}$$

Using (34)–(36), together with (19) and the definitions of the critical exponents, and taking the appropriate limits gives the desired results. ∎

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